

MATH 222 VECTOR ANALYSES

Objectives

At the end of course the student should be able:

- i. Represent vectors in 2-3 dimensions
- ii. Solve problems in vector algebra
- iii. Apply the knowledge of vector analysis in solving real life problems

Detailed scheme

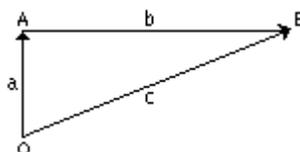
- a) Representation of vectors in 1-3 Dimensions
- b) Equality of vectors, position vectors (explain using the model of space co-ordinate)
- c) Triangular, parallelogram and polygon laws of vector addition
- d) Resultant of vectors
- e) Associative law of vectors
- f) Negative and unit vectors
- g) Magnitude or length of a vector
- h) Commutative and distribution laws of vectors, Scalar or dot product of vectors
- i) The vector or cross product of two vectors
- j) The cosine of angles between two vectors
- k) Direction cosines
- l) Relations between dot product and component of work done in a force field
- m) Triple product of vectors
- n) Plane and space curves and their vector equations
- o) Vector differentiation
- p) The grad notation
- q) The del (or vector operator notation)
- r) The divergence of a curve vector and the divergence theorem
- s) Frener –Serret formulae for solution of problems

INTRODUCTION

Vector (mathematics), quantity having both magnitude and direction. For example, an ordinary quantity, or scalar, can be exemplified by the distance 6 km; a vector quantity can be exemplified by the term 6 km north.

Vectors are usually represented by directed line segments, such as \vec{a} in the diagram below; the length of the line segment is a measure of the vector quantity, and its direction is the same as that of the vector.

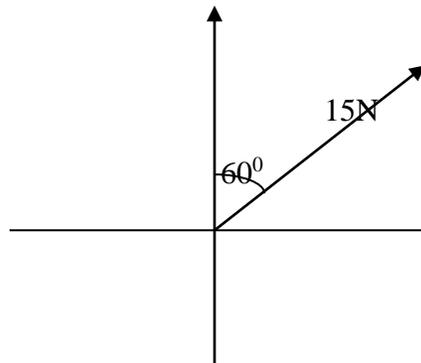
The simplest use of vectors and calculation by means of vectors is illustrated in the diagram, drawn to represent a boat moving across a stream. Vector \vec{a} , indicates the motion of the boat in the course of a given interval of time if it were moving through still water; vector \vec{b} , shows the drift or flow of the current during the same period of time. The actual path of travel of the boat under the influence of its own propulsion and of the current is represented by vector \vec{c} . By the use of vectors any type of problem involving the motion of an object being acted on by several forces can be solved graphically.



This method of problem solution, known as vector addition, is performed as follows. A vector representing one force is drawn from the origin O in the proper direction. The length of the vector is made to agree with any convenient arbitrary scale, such as a given number of centimeters to the kilometer. In the diagram the rate of rowing was 2.2 km/h, the time rowed was 1 hr, and the scale is 1 cm to 1 km. Therefore, the line \vec{a} is drawn as 2.2 cm to equal 2.2 km. The current speed of 6 km/h is then represented by a vector \vec{b} that is 6 cm long, indicating a distance of 6 km that the current moved during 1 hr. This second vector, \vec{b} , is drawn with its origin at the end of vector \vec{a} in a direction parallel to the flow of the current. The point B at the end of the second vector represents the actual position of the boat at the end of 1 hr of travel, and the actual distance traveled is represented by the length (in this case, about 6.4 km) of the vector \vec{c} .

DEFINITIONS

1. **Scalar:** It is a quantity which is specified by its magnitude only e.g. mass, speed, height, temperature, work energy, etc
2. **Vector:** It is a quantity that is specified by both its magnitude and direction and has sense e.g. displacement, velocity, force, acceleration, etc. Vectors are represented by **directed line** segments and proportionately with scale according to the size. Example a line segment \vec{AB} represents a vector whose length gives the magnitude from point A to point B and its sense is the travel from A to B. If we use \vec{e} to represent a unit vector and $|AB|$ the magnitude then we can write $\vec{AB} = |AB| \vec{e}$. Geometrically, a force of 15N acting in the direction N60°E can be represented as



3. **Unit Vector:** It is a vector whose magnitude is one unit. A unit vector in the direction of vector \vec{A} can be represented as \hat{A} . Similarly unit vectors in the direction of a vector $\vec{a}, \vec{b}, \vec{c}$ etc are also denoted by $\vec{e}_a, \vec{e}_b, \vec{e}_c$. If \vec{AB} is a vector of magnitude 4 units then $\vec{AB} = 4\vec{e}_{AB}$ i.e. the unit vector multiplied the magnitude. Since the unit vector is represented as \vec{e}_{AB} and the magnitude as 4 then $4\vec{e}_{AB}$ is the same as $|\vec{AB}| \times \vec{e}_{AB}$. In summary, every vector is equal to its magnitude $|\vec{AB}|$ multiplied by its unit vector in its direction..
4. **Zero Vector:** It is a vector whose magnitude is zero. It is also called a null vector. It is denoted by \vec{O} . Its direction and sense are of a vector specified

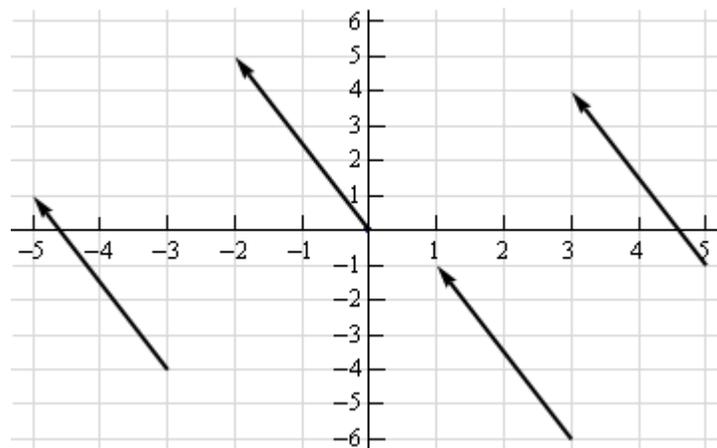
5. Negative of a vector: The negative \vec{AB} is a vector whose magnitude is the same as \vec{AB} but the direction is opposite. It is written as $-\vec{AB}$. Note that $\vec{AB} = -\vec{BA}$
6. Equal Vectors: Two vectors are equal when they have the same magnitude, have the same direction or parallel direction and same sense
7. Free Vectors: A vector whose representation is not restricted in space. Eg. A displacement of 10km to the North. It can be represented from any point.
8. Localised Vectors: A vector that must be represented along a given line or to be started from a given point eg. Force, magnetic field.
9. Proper Vectors: Any vector whose magnitude is more than zero. This means that null vectors are not proper vectors.
10. Sum or resultant of two vectors: A and B is a vector C formed by placing the initial point of B at the end point of A and then joining the initial point of A to the end point of B such that the direction of C is \vec{AC} (*this forms a triangle*).
11. Difference of Vectors A and B represented as A-B is that vector C which when added to B yields vector A. Equivalently, A-B can be defined as the sum A+(-B).
12. The product of a vector A by a scalar m is a vector mA with magnitude $|m||A|$ and with direction the same as or opposite to that of A according to the sign of m. If m is zero, the mA is a null vector

Vector Representation

Vectors only impart magnitude and direction. They don't impart any information about where the quantity is applied. This is an important idea to always remember in the study of vectors.

In a graphical sense vectors are represented by directed line segments. The length of the line segment is the magnitude of the vector and the direction of the line segment is the direction of the vector. However, because vectors don't impart any information about where the quantity is applied any directed line segment with the same length and direction will represent the same vector.

Consider the sketch below.



Each of the directed line segments in the sketch represents the same vector. In each case the vector starts at a specific point then moves 2 units to the left and 5 units up. The notation that we'll use for this vector is,

$\vec{v} = \langle -2, 5 \rangle$ and each of the directed line segments in the sketch are called **representations** of the vector.

Be careful to distinguish vector notation $\langle -2, 5 \rangle$, from the notation $(-2, 5)$ we use to represent coordinates of points. The vector denotes a magnitude and a direction of a quantity while the point denotes a location in space.

A representation of the vector $\vec{v} = \langle a_1, a_2 \rangle$ in two dimensional space is any directed line segment, \vec{AB} from the point $A = (x, y)$ to the point $B = (x + a_1, y + a_2)$. Likewise a representation of the vector $\vec{v} = \langle a_1, a_2, a_3 \rangle$ in three dimensional space is any directed line segment, \vec{AB} from the point $A = (x, y, z)$ to the point $B = (x + a_1, y + a_2, z + a_3)$.

Note that there is very little difference between the two dimensional and three dimensional formulas above. To get from the three dimensional formula to the two dimensional formula all we did is take out the third component/coordinate.

There is one representation of a vector that is special in some way. The representation of the vector $\vec{v} = \langle a_1, a_2, a_3 \rangle$ that starts at the point $A = (0, 0, 0)$ and ends at the point $B = (a_1, a_2, a_3)$ is called the **position vector** of the point (a_1, a_2, a_3) . So, when we talk about position vectors we are specifying the initial and final point of the vector. Position vectors are useful if we ever need to represent a point as a vector. Next we need to discuss briefly how to generate a vector given the initial and final points of the representation. Given the two points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ the vector with the representation \vec{AB} is, $\vec{v} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$

Note that we have to be very careful with direction here. The vector above is the vector that starts at A and ends at B . The vector that starts at B and ends at A , i.e. \vec{BA} with representation is, $\vec{w} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$

These two vectors are different and so we do need to always pay attention to what point is the starting point and what point is the ending point. When determining the vector between two points we always subtract the initial point from the terminal point.

Example 1 Give the vector for each of the following.

- (a) The vector from $(2, -7, 0)$ to $(1, -3, -5)$
- (b) The vector from $(1, -3, -5)$ to $(2, -7, 0)$
- (c) The position vector for $(-90, 4)$

Solution

(a) Remember that to construct this vector we subtract coordinates of the starting point from the ending point.

$$\langle 1 - 2, -3 - (-7), -5 - 0 \rangle = \langle -1, 4, -5 \rangle$$

(b) Same thing here

$$\langle 2-1, (-7)-(-3), 0-(-5) \rangle = \langle 1, -4, 5 \rangle$$

Notice that the only difference between the first two is the signs are all opposite. This difference is important as it is this difference that tells us that the two vectors point in opposite directions.

(c) Not much to this one other than acknowledging that the position vector of a point is nothing more than a vector with the point's coordinates as its components i.e. $\langle -90, 4 \rangle$.

Magnitude

The **magnitude**, or **length**, of the vector $\vec{v} = \langle a_1, a_2, a_3 \rangle$ is given by $\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Example 2: Determine the magnitude of each of the following vectors.

a) $\vec{a} = \langle 3, -5, 10 \rangle$

b) $\vec{u} = \langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle$

c) $\vec{w} = \langle 0, 0, 0 \rangle$

d) $\vec{i} = \langle 1, 0, 0 \rangle$

Solutions

a) $\vec{a} = \langle 3, -5, 10 \rangle \Rightarrow \|\vec{a}\| = \sqrt{9 + 25 + 100} = \sqrt{134}$

b) $\vec{u} = \langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle \Rightarrow \|\vec{u}\| = \sqrt{\frac{1}{5} + \frac{4}{5}} = 1$

c) $\vec{w} = \langle 0, 0, 0 \rangle \Rightarrow \|\vec{w}\| = \sqrt{0} = 0$

d) $\vec{i} = \langle 1, 0, 0 \rangle \Rightarrow \|\vec{i}\| = \sqrt{1} = 1$

Note that (b) and (d) are Unit Vectors

Vectors are not restricted to two dimensional or three dimensional space. Vectors can exist in general n-dimensional space. The general notation for a n-dimensional vector is, $\vec{v} = \langle a_1, a_2, a_3, a_4, \dots, a_n \rangle$ and each of the a_i 's are called **components** of the vector.

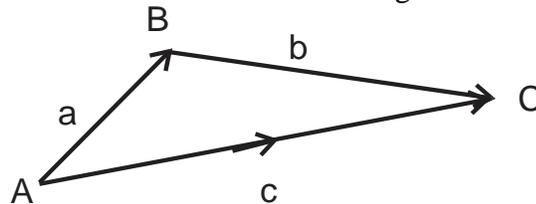
Because we will be working almost exclusively with two and three dimensional vectors in this course most of the formulas will be given for the two and/or three dimensional cases. However, most of the concepts/formulas MAT 222 notes 2016 vector analysis

will work with general vectors and the formulas are easily (and naturally) modified for general n-dimensional vectors. Also, because it is easier to visualize things in two dimensions most of the figures related to vectors will be two dimensional figures.

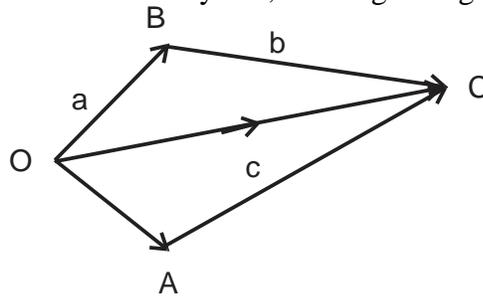
Addition of Vectors

Two vectors can be added in two different ways namely **triangle law and parallelogram law**.

1. **Triangle Law of vector addition:** Suppose a and b are two vectors respectively representing AB and BC sides of a triangle ABC according to a given scale, the addition of vectors a and b represented as $a + b$ is defined by the vector whose magnitude is the side AC of the triangle and direction \vec{AC}



2. **Parallelogram law of vector addition:** Suppose a and b are two vectors localized from the point O and representing the lines OA and OB . Construct vector BC equal to OA and vector AC equal to OB . Then the sum of a and b written as $a + b$ is defined by OC , the longer diagonal of parallelogram $AOBC$.



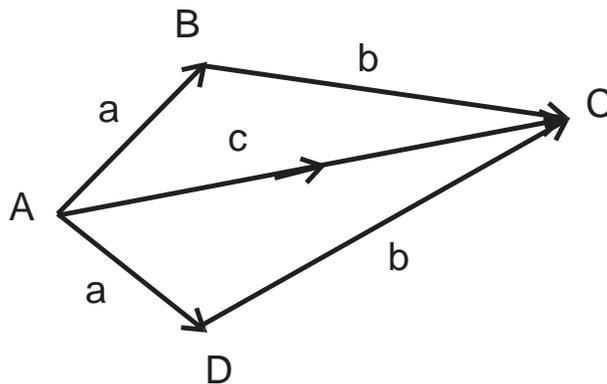
Commutative Law of Vector addition

If a and b are two vectors, then

$$a + b = b + a$$

Proof:

Using the triangle law, draw triangle ABC such that $AB = a$ and $BC = b$.

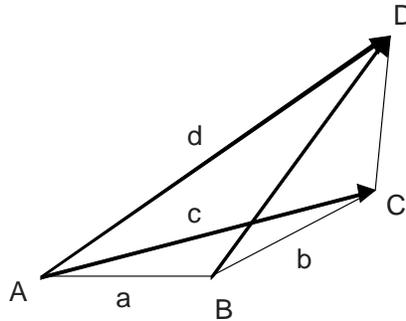


By triangular law, $AB + BC = AC$ i.e. $a + b = AC$ (1)

Next Complete the parallelogram ABCD such that
 $AD = BC = b$ and $AB = DC = a$
 Then in Triangle ADC, $AD + DC$ i.e. $b + a = AC$ (2)
 From (1) and (2), $a + b = b + a$.

Associative law of Vector addition

Let a, b, and c be any three vectors then $(a+b)+c=a+(b+c)$



Using Triangle law, Let $AB=a$, $BC=b$ and $CD=c$

Proof:

By the triangle law of vector addition, $a + b = AC$

$\therefore (a+b)+c=AC+CD=AD$ (1)

In triangle BCD, $b+c=BD$

$\therefore a+(b+c)=AB+BD=AD$ (2)

Thus from (1) and (2)

$(a+b)+c=a+(b+c)=AD$

Thus Proving that addition of vectors is associative.

Transposition of Vectors

As in algebra, vectors can be transferred from one side of an equation to the other. Hence if a,b and c are vectors such that $a-b=c$ then we can write

$a = b + c$

Proof:

Let the sides of triangle AB, CB and AC respectively represent vectors a,b and c. We find that $AB+BC=AC$ i.e. $a + (-b) = c$ since $CB = b$

$\Rightarrow a-b = c$ (1)

Also $AB = AC + CB$

$\Rightarrow a = c + b$ (2)

From (1) and (2) it is clear that we can transpose vectors from one side to the other

Addition of vectors by inspection

Vectors can be added by inspection only when they have a common point then we employ the transposition and rearrangement of the letters and the sum becomes the remaining letters. For instance, the sum of vectors AB and BC is AC.

Example: Find the sum of the vectors; AB, DC,DA and 2DC.

Solution: $AB+(-CD)+DA+CD+CD$

$\Rightarrow AB-CD+DA+CD+CD$

$\Rightarrow AB+DA+CD \dots$

Vector Equation of a Triangle ABC

This is given by: $AB + BC + CA = 0$
 Proof: $AB + BC = AC$ by inspection
 Subtract AC from both sides $\Rightarrow AB + BC - AC = 0$
 But $-AC = CA$
 Hence $AB + BC + CA = 0$

Vector Equation of a quadrilateral ABCD

This is given by $AB + BC + CD + DA = 0$
 This equation implies that is 3 sides of a quadrilateral are vectors, the 4th side in the opposite sense is the sum of the first three. For instance, the sum of $AB + BC + CD = AD \dots$ (by transposition)(1)
 Thus $AB + BC + CD - AD = 0$
 But $AD = -DA$ hence $AB + BC + CD + DA = 0$

Addition of Several Vectors

Suppose $a_1, a_2, a_3, \dots, a_n$ are n vectors represented by sides of a polygon say $A_0 A_1, A_1 A_2, A_2 A_3, \dots, A_n A_0$ then their sum is given by $A_0 A_n$ which is called the resultant of the vector and the polygon is called vector polygon.

Components of a vector

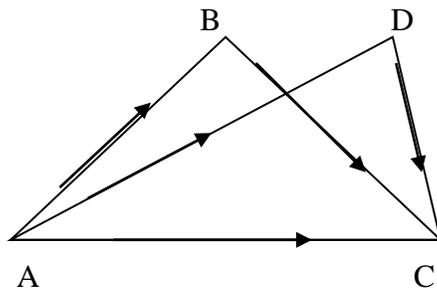
If vector c is the sum of vectors a and b , then vectors a and b are called the components of vector c in their particular direction i.e.

$$c = a + b$$

Thus from the components of AC in the diagram below are respectively are as follows;

$$AC = AB + BC \text{ and}$$

$$AC = AD + DC$$



Examples

1. Show that the moduli of the components of vector r which are inclined at angles 30° and 60° to r are

$$\frac{\sqrt{3}}{2}|r| \text{ and } \frac{1}{2}|r|$$

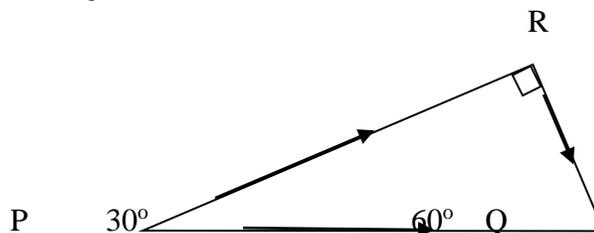
Solution:

Represent the info in a diagram (say triangle PQR) such that $PQ = r$ and the sides PR and QR will incline resp. to PQ at 30 and 60 respectively from P and Q .

Since PR and QR are the components of PQ then

$$PQ = PR + RQ.$$

Here we note that the magnitude of $PR = RP$ and from the ensuing right angled triangle



$$\cos 30^\circ = \frac{PR}{PQ}$$

$$\Rightarrow PR = PQ \cos 30^\circ = \frac{\sqrt{3}}{2}|r| \text{ where } \cos 30 =$$

By the same token, $QR=PQ\cos60^\circ = \frac{1}{2}|r|$

2. If $BC=BA+PQ-CB$, show that AB parallel to PQ

Solution:

(note that if two vectors are equal, then they are either parallel if they have no common point or they are collinear if they are localized to a common point)

Thus we have to show that AB and PQ are equal.

$$\begin{aligned} \text{Hence; } BC=BA+PQ-CB &\Rightarrow BC+CB=BA+PQ \dots\dots \text{ by transpose of vectors} \\ &\Rightarrow BC-BC=BA+PQ \\ &\Rightarrow BA+PQ=0 \\ &\Rightarrow PQ= -BA= AB \dots\dots \text{ as required} \end{aligned}$$

3. Prove that P coincides with R if $PQ+PS=RQ+RS$

Solution:

(If two points coincide, then their magnitude is zero) thus we only have to show $PR=0$

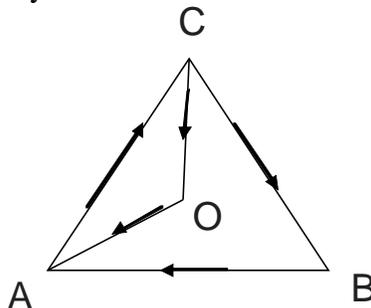
$$\begin{aligned} \text{Hence if } PQ+PS=RQ+RS &\Rightarrow PQ+QR = RS+SP \dots\dots \text{ transposition and use of negative of vectors} \\ &\Rightarrow PR=RP \\ &\Rightarrow PR+PR=0 \dots\dots \text{ Transposition and use of negative of vectors} \\ &\Rightarrow 2PR=0 \\ &\Rightarrow PR=0 \end{aligned}$$

Therefore P coincides with R .

4. If O is any point within ΔABC , Prove that $OA+CO=CB+BA$

Solution:

Represent the information diagrammatically



In ΔAOC ; $CO+OA=CA \dots\dots\dots(1)$

In ΔABC ; $CB+BA=CA \dots\dots\dots(2)$

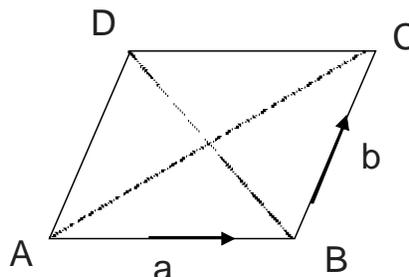
Therefore $CO+OA= CB+BA=CA$

Therefore; $OA+CO=CB+BA$

5. ABCD is a parallelogram. If $AB=a$ and $BC=b$, show that $AC=a+b$ and $BD=b-a$

Solution:

From the diagram,



In $\triangle ABC$, $AB + BC = AC \Rightarrow a + b = AC$ as reqd
 But opposite sides of a parallelogram are equal
 Therefore, $AD = BC = b$
 $\triangle ABD$, $AB + BD = AD \Rightarrow a + BD = b$
 $\Rightarrow BD = b - a$ as reqd

6. If $AO+OB = BO+OC$, prove that A,B,C are collinear (ie. They lie on the same line) and $AB=BC$.

Solution:

$$AO+OB=BO+OC$$

$$AB=BC.$$

7. ABCD is a quadrilateral with $AB=DC$. Prove that ABCD is a parallelogram.

Solution:

Since AB and DC are equal and not linked by any common point, they are parallel.

Next, for a quadrilateral ABCD to be a parallelogram, AD must be equal to BC

From $\triangle ADC$,

$$AD + DC = AC$$

$$\Rightarrow AD = AC - DC$$

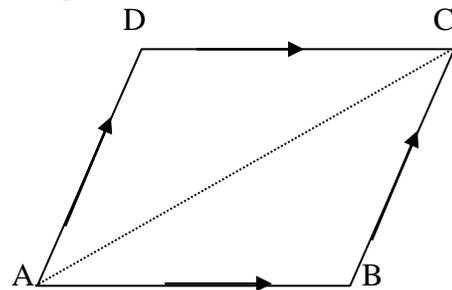
$$\text{But } DC = AB$$

$$\text{Thus } AD = AC - AB$$

$$= AC + BA$$

$$= BA + AC \text{ (by commutative law)}$$

Therefore $AD = BC$



8. Prove that a quadrilateral whose diagonals bisect each other is a parallelogram.

Solution:

Let the quadrilateral be ABCD and let the point of intersection of the diagonals be O such that $AO=OC$ and $BO=OD$.

From $\triangle AOB$, $AO + OB = AB$

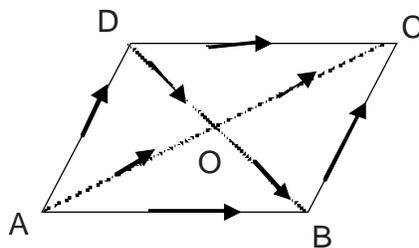
From $\triangle DOC$, $DO + OC = DC$.

But $AO + OB = DO + OC$ (Bisecting diagonals as given).

Therefore, $AB = DC$

Similarly from $\triangle AOD$ and $\triangle BOC$, we can show by the same token that $AD=BC$

Hence the quadrilateral ABCD is a parallelogram.



Using the Coordinate Plane in Vector Calculations

Using real numbers, we have been able to associate each point on a graph shown on a plane (O, x) with a number called its x coordinate.

Likewise, we can associate each point on a plane (O, x, y) with two numbers that are the coordinates of the point.

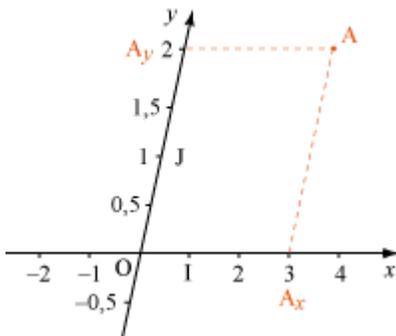
On a plane we can calculate the coordinates of a vector and carry out different types of vector analysis in order to solve geometry problems.

1. How do we locate a point on a plane?

Let's begin by defining a plane. A plane is a trio of noncollinear points (the word *trio* means that the three points under consideration are ordered).

In general, we call the plane (O, x, y) , where O is the origin of the plane; the line (Ox) is the horizontal axis and the line (Oy) is the vertical axis.

Next, using the plane, we associate a point with a pair of real numbers by drawing lines parallel to the axes and passing through the point.



For example, let's find the coordinates of A on the diagram above.

The point where (Ox) and the line parallel to (Oy) , passing through A , meet is called A_x and the point where (Oy) and the line parallel to (Ox) , passing through A , meet is called A_y .

To work out the coordinates of A :

For the x coordinate of A , take the value of the point A_x on the x -axis (Ox) starting at O ;

For the y coordinate of A , take the value of the point A_y on the y -axis (Oy) starting at O ;

In this case the coordinates of the point A are $(3, 2)$.

Notes

If the axes are perpendicular (O, x, y) is a right-angled xy -plane.

2. How do we define a vector? When are two vectors equal?

Given a plane on which we have defined a unit of length, a vector \overrightarrow{AB} is characterized in three ways:

By the direction of the line (AB) ;

By its sense: A towards B ;

And by its length: the distance AB .

The vector \overrightarrow{AB} is equal to the vector \overrightarrow{CD} if these two vectors have:

The same direction, i.e. $(AB) \parallel (CD)$;

The same sense, which means that points B and D are at the same end of the line (AC) ;

The same length, which means that $AB = CD$.

In other words $\overrightarrow{AB} = \overrightarrow{CD}$ if and only if $ABDC$ is a parallelogram.

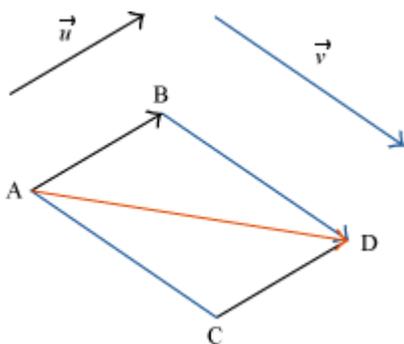
Also:

$\overrightarrow{AB} = \overrightarrow{CD}$ if and only if the reflection of C through translation of A to B is D .

$\overrightarrow{AB} = \overrightarrow{CD}$ if and only if $[AD]$ and $[BC]$ have the same midpoint.

3. What operations can we carry out on vectors?

The sum of two vectors is a vector that can be made using the parallelogram rule: $\vec{u} + \vec{v} = \overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$.



The multiplication of a vector by a real number is defined as follows:

\vec{u} is a vector other than zero and k a real number other than zero, the vector $\vec{v} = k\vec{u}$ is defined as follows:

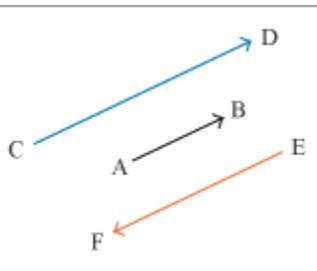
\vec{v} has the same direction as \vec{u} ;

\vec{v} has the same sense as \vec{u} if k is positive, the opposite is true if k is negative. If $k = -1$, then $-1 \times \vec{u} = -\vec{u}$, this defines the vector that is opposite to \vec{u} .

Collinear vectors are vectors that have the same direction. Vectors \vec{u} and \vec{v} are collinear if and only if there is a real number k such that $\vec{v} = k\vec{u}$.

Example: In the figure on the right, we have

$\vec{CD} = 2.5\vec{AB}$ and $\vec{EF} = -2\vec{AB}$. Vectors \vec{AB} , \vec{CD} and \vec{EF} have the same direction therefore they are collinear.

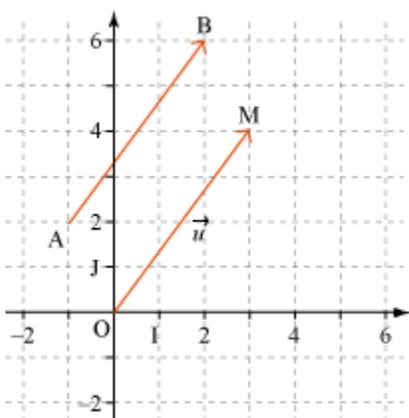


4. What is the basis of vector analysis?

On a coordinate plane (O, x, y) , any vector \vec{u} is associated to a unique point M such that $\vec{OM} = \vec{u}$. The point M is the image of the origin O of the plane through translation of the vector \vec{u} .

By definition, the coordinates of \vec{u} are those of M . If M has the coordinates (x, y) , the vector \vec{u} has the

coordinates (x, y) . This is written $\vec{u}(x, y)$ or $\vec{u} \begin{vmatrix} x \\ y \end{vmatrix}$. For example, in the diagram below $\vec{u}(3, 4)$.



It follows that two vectors $\vec{u}(x, y)$ and $\vec{v}(x', y')$ are equal if and only if they have the same coordinates: $x = x'$ and $y = y'$.

It is possible to work out the coordinates of any vector \vec{AB} given the coordinates of the points A and B . On a plane, if A has the coordinates (x_A, y_A) and B has the coordinates (x_B, y_B) , then the vector \vec{AB} has the coordinates $(x_B - x_A, y_B - y_A)$.

If \vec{u} and \vec{v} are two vectors with the coordinates $\vec{u}(x, y)$ and $\vec{v}(x', y')$, then:

The sum of the two vectors $\vec{u}(x, y)$ and $\vec{v}(x', y')$ is vector $\vec{u} + \vec{v}$ with coordinates $(x + x', y + y')$;

The product of vector $\vec{u}(x, y)$ and a real number k is vector $k\vec{u}$ with coordinates (kx, ky) .

Two vectors have the coordinates $\vec{u}(x, y)$ and $\vec{v}(x', y')$.

The collinear nature of vectors \vec{u} and \vec{v} is expressed as two equations:

$\vec{v} = k\vec{u}$ if and only if $x' = kx$ and $y' = ky$.

A simpler way of expressing this property is the “cross-multiplication” rule of proportionality:

\vec{u} and \vec{v} are collinear if and only if $xy' = x'y$.

For example, the vectors $\vec{u} \begin{vmatrix} 3 \\ -2 \end{vmatrix}$ and $\vec{v} \begin{vmatrix} -7.5 \\ 5 \end{vmatrix}$ are collinear because $3 \times 5 = (-2) \times (-7.5) = 15$.

If A and B are two points whose coordinates are (x_A, y_A) and (x_B, y_B) respectively, then this gives us

$$\|\overline{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

Reminder:

A plane is a trio of noncollinear points. We associate each point on the plane with two real numbers, its coordinates, by drawing the lines parallel to the axes and passing through this point.

On a plane showing a unit of length, a vector \overline{AB} is characterized in three ways: its direction, its sense, and its length.

The sum of two vectors $\vec{u}(x, y)$ and $\vec{v}(x', y')$ is vector $\vec{u} + \vec{v}$ with coordinates $(x + x', y + y')$. The product of vector $\vec{u}(x, y)$ and a real number k is vector $k\vec{u}$ with coordinates (kx, ky) .

The vectors $\vec{u}(x, y)$ and $\vec{v}(x', y')$ are collinear if and only if $xy' = x'y$.

Calculating the Coordinates of a Vector and the Midpoint of a Line Segment

The coordinates of a vector \vec{u} can be interpreted by a translation where we have chosen \overline{AB} as a representative of this vector. What relationship connects the coordinates of \vec{u} and those of A and B? From this relationship, how can we calculate the coordinates of the midpoint of a segment if we know its extremities?

I. Calculating the coordinates of a vector

1. Calculation formula

(O, I, J) is an x-y plane, and A (x_A, y_A) and B (x_B, y_B) are two points.

The coordinates of vector \overline{AB} are given by the formula $\overline{AB}(x_B - x_A, y_B - y_A)$.

Example: (O, I, J) is an x-y plane and A $(2, -4)$, B $(-3, -1)$ are two points.

Calculate the coordinates of vector \overline{AB} .

By applying the formula, we can write $\overline{AB}(-3-2, -1-(-4))$, so \overline{AB} is $(-5, 3)$.

We can check these coordinates by reading the graph, then plotting points A and B:

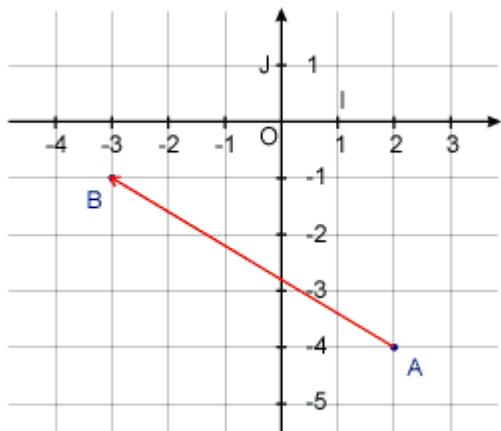


Figure 1

2. Application

Statement: (O, I, J) is an x-y plane. Plot points E (-3, 1), F (3, 5), G (4, 2), and H (-2, -2) and prove that quadrilateral EFGH is a parallelogram.

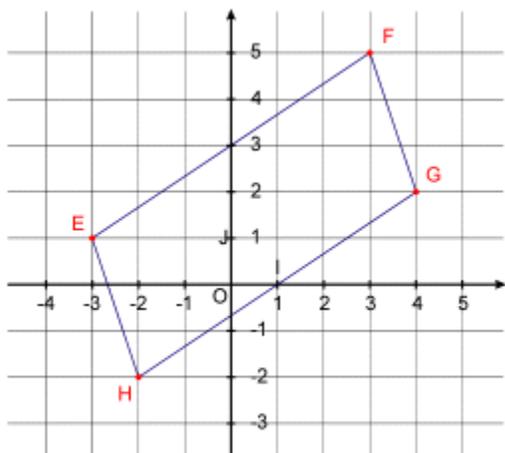


Figure 2

Solution: We simply have to prove the vectorial equation $\overrightarrow{EF} = \overrightarrow{HG}$. To do this, calculate the coordinates of these two vectors.

\overrightarrow{EF} (3-(-3), 5-1), so \overrightarrow{EF} (6, 4).

\overrightarrow{HG} (4-2(-2), 2-(-2)), so \overrightarrow{HG} (6, 4).

Vectors \overrightarrow{EF} and \overrightarrow{HG} have the same coordinates.

We accept that two vectors with the same coordinates are equal.

Therefore, $\overrightarrow{EF} = \overrightarrow{HG}$, so quadrilateral EFGH is a parallelogram.

II. Calculating the coordinates of the midpoint of a segment

1. Calculation formula

(O, I, J) is an x-y plane and A (x_A , y_A), B (x_B , y_B) are two points. If M is the midpoint of segment [AB], then

$$M\left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}\right)$$

Proof: If M is the midpoint of [AB], then $\overrightarrow{AM} = \overrightarrow{MB}$. Vectors \overrightarrow{AM} and \overrightarrow{MB} have the same direction, as A, M, and B are collinear. They have the same length, since MA = MB. Therefore, these two vectors are equal.

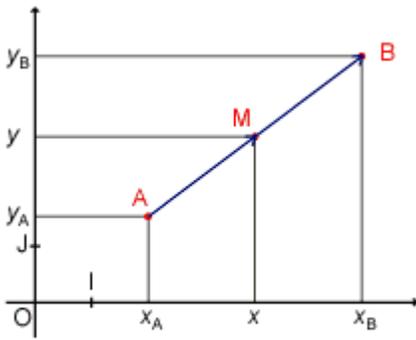


Figure 3

Let's call the coordinates of M (x, y) , and write out the coordinates of \overrightarrow{AM} and \overrightarrow{MB} :

$\overrightarrow{AM}(x - x_A, y - y_A)$ and $\overrightarrow{MB}(x_B - x, y_B - y)$.

Since vectors \overrightarrow{AM} and \overrightarrow{MB} are equal, we can denote that their coordinates are equal. Therefore, we find that $x - x_A = x_B - x$ and $y - y_A = y_B - y$.

These two equations are equivalent to:

$$2x = x_A + x_B \text{ and } 2y = y_A + y_B, \text{ so } x = \frac{x_A + x_B}{2} \text{ and } y = \frac{y_A + y_B}{2}.$$

Therefore, we have $M\left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}\right)$.

Example: (O, I, J) is an x-y plane and U $(-3, 2)$ and T $(5, 4)$ are two points. Calculate the coordinates of midpoint H of segment [UT].

By applying the above formula, we can write $H\left(\frac{-3+5}{2}, \frac{2+4}{2}\right)$, from which we find H $(1, 3)$.

We can check this calculation by plotting the points on the grid (O, I, J) .

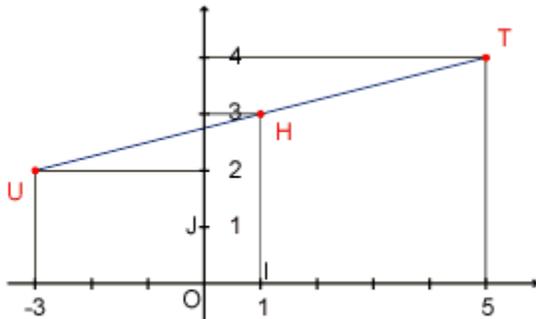


Figure 4

2. Application

The formula for the coordinates of the midpoint of a segment gives us an alternative way of proving that a quadrilateral is a parallelogram.

Statement: (O, I, J) is an x-y plane. Plot points K $(-4, -1)$, L $(-2, 3)$, M $(6, 5)$, and N $(4, 1)$, and prove that quadrilateral KLMN is a parallelogram.

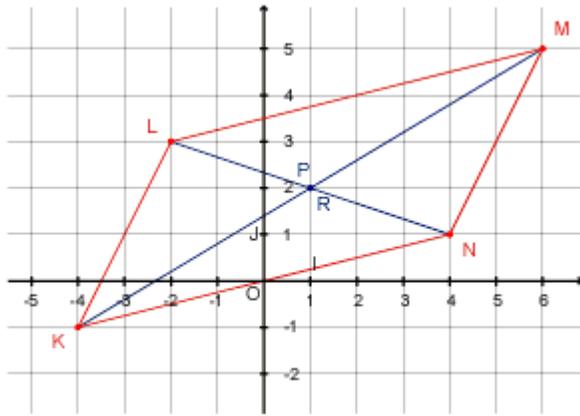


Figure 5

Solution: We will prove that segments $[KM]$ and $[LN]$ have the same midpoint. To do this, let's call P the midpoint of $[KM]$ and R the midpoint of $[LN]$ and calculate the coordinates of these two points:

$$P\left(\frac{-4+6}{2}, \frac{-1+5}{2}\right), \text{ therefore } P(1, 2).$$

$$R\left(\frac{-2+4}{2}, \frac{3+1}{2}\right), \text{ therefore } R(1, 2).$$

As points P and R have the same coordinates, they are coincident or the same point. From this, we can work out that segments $[KM]$ and $[LN]$ have the same midpoint.

The diagonals of quadrilateral $KLMN$ have the same midpoint. Therefore, this quadrilateral is a parallelogram.